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Banach $*$-algebras, B^*-seminorms, and positive functionals

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Abstract

This paper concerns Banach $*$-algebras which are nonunital or have bounded approximate identity. A necessary and sufficient condition is given for a B^*-seminorm to be regular. A one-to-one correspondence between the carrier space of a complex Banach $*$-algebra and the set of all regular B^*-seminorms is established. Interaction between α-bounded functionals and regular B^*-seminorms is examined. We prove that the carrier space of a Banach $*$-algebra is identical with the set of all extreme points of positive linear functionals. The main approach is via an ordering of the algebra, the positive cone being the closure of the set of certain elements in the set of self-adjoint (hermitian) elements of the algebra (but not in the algebra) where the involution on the algebra is not continuous. A characterization of the set of centralizers for algebra in terms of positive cones where the involution is not continuous is given. This result improves some previous results on this topic. In the context of approximate identity of norm less than or equal to the real number 1, we prove that a cone is closed with respect to a special product if and only if the algebra is commutative modulo its radical. An application of the Shirali–Ford theorem is also discussed. In particular, the equality between the two positive cones is established if the algebra is symmetric.

Keywords: B^*-seminorms; Banach $*$-algebras; Extreme points; α-Bounded functionals

1. B^*-Seminorms

Let A be a Banach $*$-algebra. A B^*-seminorm on A is a function $\eta : A \to R$ such that for all a, b in A and $z \in C$, the following are true:

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$$\eta(a+b) \leqslant \eta(a)+\eta(b),$$
$$\eta(za) = |z|\eta(a),$$
$$\eta(ab) \leqslant \eta(a)\eta(b),$$
$$\eta(a^*a) = \big(\eta(a)\big)^2.$$

The fundamental reference to these norms is [4]. In fact, a B^*-seminorm on A is an algebra seminorm η on A such that $\eta(a^*a) = (\eta(a))^2$for all a in A. The set of all B^*-seminorms on A is denoted by $\wp(A)$.

A B^*-seminorm η on A is a regular seminorm if it is a member of every subset P of $\wp(A)$ such that for all a in A, $\eta(a) = \max\{\mu(a)\colon\ \mu \in P\}$. The set $\Re(A)$ denotes the set of all regular B^*-seminorms on A.

An element b of A has the property

$(*)$ if $\mu = \eta$ whenever μ is a member of $\Re(A)$ satisfying the ordering $\mu \leqslant \eta$, $\mu(b) = \eta(b)$.

It is clear that $\eta \in \Re(A)$ if an element in A has $(*)$-property. We also observe that the set $\wp(A)$ is compact in the topology of pointwise convergence.

Let $\alpha(a) = \sup\{p(a)\colon\ p \in \wp(A)\}$. Then $\alpha(a)$ is a B^*-seminorm on A. Also, it is the greatest B^*-seminorm on A in the pointwise ordering.

A linear functional f on A is α-bounded if there exists a positive constant M_f such that $|f(a)| \leqslant M_f\alpha(a)$, $\forall a \in A$. We emphasize the dependence of M on f. Among the properties of these functionals, an important one is its continuity with respect to the original norm on A. If $D(\alpha)$ denotes the set of all α-bounded functionals then it is a subspace of A^*, the dual space of A. Moreover, $|f|_\alpha = \sup\{|f(a)|\colon\ \alpha(a) \leqslant 1,\ a \in A\}$.

It is not hard to check that $|f(a)| \leqslant |f|_\alpha \alpha(a)$, $\forall a \in A$. A positive α-bounded linear functional f on A is a state if $|f|_\alpha = 1$.

Example 1.1. An interesting application of α-bounded functionals can be seen in α-bounded operators.

Let E be a locally m-convex algebra with unit e and let $\{q_m\}$ be a separating family of sub multiplicative seminorms which generates the topology and is such that $q_m(e) = 1$, $\forall m$. Given $(E, \{q_m\})$, let N_m denote the null space of q_m and E_m the quotient space E/N_m. For each m, consider the natural mapping $a \to a_m$ of E onto E_m. For each m, E_m is a normed algebra with identity e_m with $\|e_m\|_m = 1$ and norm defined by $\|a_m\|_m = q_m(a)$.

For a given E we say a linear operator T on E is α-bounded if for all m there exists $\alpha_m > 0$ such that $q_m(Ta) \leqslant \alpha(m)q_m(a)$.

Let E be the algebra of complex valued map on C, the set of all complex numbers. If C_0 is a complex subset of C then the topology on E is given by the following family of seminorms, $\{q_{C_0}\}$: $q_{C_0}(f) = \max\{|f(z)|\colon\ z \in C_0\}$.

Define a linear operator T on E by $(Tf)(z) = zT(z)$, $\forall z \in C$. In this case we have $q_m(Tf) = |z| \cdot \|f(z)\| = |z|q_m(f)$ and by the definition of α-bounded operators, the operator T belongs to $G = G(E, \{q_m\})$, the set of all α-bounded operators on E. The set G is a sub-algebra of the algebra of all continuous linear operators on E.

Now let $r_m = \sup\{q_m(Tf)\colon\ q_m(f) \leqslant 1,\ \forall f \in E\}$. Then the family $\{r_m\}$ is a separating family of sub-multiplicative seminorms on G and $r_m(I) = 1$, $\forall m$, where I is the identity operator. The algebra G with the topology generated by this family of sub-multiplicative seminorms is a unital locally m-convex algebra.

If $q_m(f) \leqslant \alpha(m)q_m(f)$ then the set of such operators is a sub-algebra of G where $\|T\| = \sup_m r_m(T)$.

Lemma 1.2. *If e is the unit of A, then $|f|_\alpha = f(e)$.*

Proof. The inequality $|f|_\alpha \geqslant f(e)$ holds since $\alpha(e) \leqslant 1$. An application of the Cauchy–Schwarz inequality yields that $|f(a)|^2 \leqslant f(e)f(a^*a)$, $\forall a \in A$. This implies that $|f(a)|^2 \leqslant f(e)|f|_\alpha \alpha(a^*a) = f(e)|f|_\alpha(\alpha(a))^2$. Thus, $(|f|_\alpha)^2 \leqslant f(e)|f|_\alpha$. □

Remark 1.3. By Theorem 2.2 of [9] and Lemma 1.2, we have $\|f\| = f(e) = |f|_\alpha$. Also, since the spectral radius function $\rho(\cdot)$ is sub-multiplicative on commutative algebras, we have $\alpha(a) \leqslant (\rho(a^*))(\rho(a))^{1/2}$, $\forall a \in A$. If in addition, the involution on A is hermitian, the function $\rho(\cdot)$ is a B^*-seminorm on A. Hence, $\rho(a) \leqslant \alpha(a)$, $\forall a \in A$. Thus, on a commutative Banach $*$-algebra with the hermitian involution, the greatest B^*-seminorm coincides with the spectral radius.

A positive linear functional f is a pure state of A if it is nonzero and α-bounded. Moreover, if any α-bounded linear functional is dominated by f then it is of the form βf with $0 \leqslant \beta \leqslant 1$.

Example 1.4. Let f be a positive α-bounded linear functional on A and suppose that Π_f is the associated $*$-representation. Then for all a in A,

$$\eta_f(a) = \left|\Pi_f(a)\right|.$$

The functional f is representable by Theorem 3.1 of [9] and η_f is a B^*-seminorm on A.

The following proposition proposes a condition on η so that it becomes a pure state of A.

Proposition 1.5. *Let η be a regular B^*-seminorm on A. Then there exists a nonzero topologically irreducible star representation Π on A such that for all elements a in A, $\eta(a) = |\Pi(a)|$.*

Proof. Since $\alpha(a) = \sup\{p(a)\colon\ p \in \wp(A)\}$ is the greatest B^*-seminorm on A in the pointwise ordering and η is nonzero, from Theorem 3.1 of [8] it follows that it is a pure state of A. The proof is complete by Corollary 2.1 of [8]. □

2. One-to-one correspondence between the sets A_h and $\mathfrak{R}(A)$

A complex homomorphism on A is a nonzero linear functional f on A satisfying $f(xy) = f(yx)$, $\forall x, y \in A$. An involution is said to be hermitian if the spectrum of every self-adjoint element of the algebra is a subset of positive real line. The set of all hermitian complex homomorphism on A is denoted by A_h. Let $\hat{A}$ be the carrier space of A and its elements are called the characters of A. We remark that the involution is hermitian if and only if $A_h = \hat{A}$.

In what follows, we assume neither the continuity of the involution on A nor that the involution is hermitian.

Before we establish the correspondence between the sets A_h and $\Re(A)$ to be one-to-one, we prove the following proposition.

Proposition 2.1. *Let $f \in A_h$. Then f is a positive linear α-bounded functional on A such that $|f|_\alpha = 1$.*

Proof. Let $f(a^*a) = |f(a)|^2 \geqslant 0$, $\forall a \in A$. Then f is a positive functional. We note that $|f| = \alpha(a)$, $\forall a \in A$, and hence f is α-bounded and $|f|_\alpha = 1$. This follows by Remark 1.3. □

Theorem 2.2. *There exists a one-to-one correspondence between the sets A_h and $\Re(A)$ via the map $f \to |f|$.*

Proof. Let $f \in A_h$. Then by Proposition 2.1 we have $|f| \in \wp(A)$ because for a given $\eta \in \wp(A)$ and for all elements a in A

$$\eta_f(a) = \sup\{f(x^*a^*ax)^{1/2}\colon\ f(x^*x) \leqslant 1\} = |f(a)|, \quad x \in A.$$

Next, we will assert that $|f| \in \Re(A)$. Let η be a B^*-seminorm on A such that for all $b \in A$, $f \neq 0$, $\eta \leqslant |f|$, and $\eta(b) = |f(b)|$. Since $\eta \in \wp(A)$ and $b \in A$, there exists a positive linear functional ψ on A such that for all a in A,

$$\eta(b^*b) = \psi(b^*b) \quad \text{and} \quad \eta(a) \geqslant |\psi(a)|.$$

Hence, for all a in A,

$$\eta(b^*b) = \psi(b^*b) \quad \text{and} \quad f(a^*a) \geqslant \psi(a^*a).$$

Now we show that $\psi = f$. Clearly, f is a pure state of A an d ψ is a α-bounded functional. Thus we may assume that $\psi = \beta f$, $\beta \in [0, 1]$. We claim that $\beta = 1$. To show this, let us consider $\beta f(b^*b)$. Then

$$\beta f(b^*b) = \eta(b^*b) = \psi(b^*b) = (\eta(b))^2 = |f(b)|^2 = f(b^*b)$$
$$\Rightarrow \quad \beta = 1 \quad \Rightarrow \quad \psi = f.$$

Hence, $\eta \geqslant |f|$ so that $\eta = |f|$. In fact, we have shown that the element b in A has the $(*)$-property. This means that $f \in \Re(A)$. Suppose that ψ is a pure state of A and $\eta \neq 0$. Then, $\psi \in A_h \Rightarrow \eta = |\psi| = \eta_\psi$. Finally, suppose that $|f_1| = |f_2|$ for $f_1, f_2 \in A_h$. Then for all x in A, $f_1(x^*x) = f_2(x^*x)$. We recall the following identity:

$$4xy = (y + x^*)^*(y + x^*) - (y - x^*)^*(y - x^*) + i(y + ix^*)^*(y + ix^*) - i(y - ix^*)^*(y - ix^*).$$

An application of the above identity yields the equality $f_1(xy) = f_2(xy)$, $\forall x, y \in A$.

Further, if x is a self-adjoint element of the algebra A then

$$(f_1(x))^3 = f_1(x^3) = f_1(x^2x) = f_2(x^2x) = (f_2(x))^3 = f_2(x^3).$$

This implies that for all self-adjoint elements x of A, $f_1 = f_2$ and the correspondence is one-to-one. □

3. Applications of Theorem 2.2

Here we present the existence of a regular B^*-seminorm on A and an extreme point associated with this seminorm.

Theorem 3.1. *Let Π be a topologically irreducible $*$-representation of a commutative algebra A over a Hilbert space H. Then there exists a regular B^*-seminorm η on A such that for all elements a in A, $\eta(a) = |\Pi(a)|$.*

Proof. The commutant of $\Pi(a)$ in $B(H)$ is the set of all scalar multiples of the identity operator I. Since the algebra A is commutative, it follows that there exists a scalar λ such that for all elements a in A, $\Pi(a) = \lambda I$. For each a in A, let $\lambda = f(a)$. If Π is a nonzero $*$-representation of A, then $f \in \hat{A}$ and $|f(a)| = |\Pi(a)| = |\lambda|$, $\forall a \in A$. The proof follows from Theorem 2.2. □

Remark 3.2. Combining Proposition 1.5 and Theorem 3.1, we obtain a necessary and sufficient condition for a B^*-seminorm to be regular.

Theorem 3.3. *Let $\eta \in \wp(A)$ and $\theta_n = \{u \in D(\alpha)\colon\ \eta(a) \geqslant |u(a)|,\ a \in A\}$. If $b \in A$ then there exists an extreme point u of θ_n: $u(b^*b) = \eta(b^*b)$.*

Proof. Let $U = \{u \in \theta_n\colon\ u(b^*b) = \eta(b^*b)\}$. Then U is nonempty weak $*$-compact and convex set. By the Krein–Milman theorem the sets U and θ_n have the same extreme point u. □

The above discussion on B^*-seminorm and Theorem 3.1 are the motivations for the next work on commutativity of Banach $*$-algebras.

4. Cones in Banach $*$-algebras

An element a in A is called hermitian if its numerical range is a subset of positive real numbers. Since the algebra numerical range is the best approximant of the spectrum, a hermitian element of the algebra will have a positive spectrum. Let $H(A)$ denote the set of all hermitian elements of A. For a Banach algebra with unit e, the set $H(A)$ is a real Banach space. Moreover, if $K(A)$ is the set of all positive elements of A then $K(A)$ is a generating cone in $H(A)$. If the algebra has unit and that is in the interior of $K(A)$ then the unit in this case is an order unit.

Let A be a complex normed algebra such that $A = H(A) + iH(A)$ and the left regular representation is faithful. Then A is a pre-B^*-algebra with the operator norm given by the left regular representation. If A is complete then A is an A^*-algebra. Moreover, if $\|\cdot\|$ is the Banach algebra norm on A, then A is a B^*-algebra if and only if

$$\|a\| = \sup\bigl\{\|ax\|\colon\ x \in A,\ \|x\| \leqslant 1\bigr\}, \quad a \in A. \tag{I}$$

It is easy to verify that if A has an approximate identity of norm less than or equal to 1 then A is a B^*-algebra. In the following result, we describe the relationship between the sets $H(A)$ and $K(A)$.

Theorem 4.1. *Let A be a Banach algebra where the norm of elements is given by* (I). *Then $K(A)$ is a closed, normal, and generating cone in $H(A)$.*

Proof. From the definition of positive elements we have $K(A) = K(A) \subset K(A)$ and $R^+K(A) \subseteq K(A)$. Here $R^+ = [0, \infty)$.

First, consider the case when A has an identity e of norm one. Then $K(A)$ is a closed and normal cone in $H(A)$ with e in its interior. Hence, $K(A)$ generates $H(A)$.

Next, we prove that $K(A)$ is normal. Suppose that A has no unit. Then for a and b in $K(A)$, the numerical ranges of a and b are real. Let $D_A(x) = \{f \in A'\colon\ \|f\| = 1 = f(x)\}$. Then for each f in this set, $f(ac) \geqslant 0$, $f(bc) \geqslant 0$, and $f(ac) + f(bc) \geqslant f(ac)$. This implies that the numerical radius $v(a) \leqslant \sup\{|\lambda|\colon\ \lambda \in V_c(a+b)\}$ where $\lambda = f(ac) + f(bc)$ and $V_A(a) = \bigcup_{\|b\|=1}\{\mathring{V}_b(A, a)\}$. The set $\mathring{V}_b(A, a)$ is the relative numerical range of the element a related to the element b. By this reasoning and the fact that the numerical radius is a monotone norm on A, we have the cone $K(A)$ is normal.

To prove that the cone $K(A)$ is closed, let $\{a_n\}$ be a sequence of positive elements of A such that $a_n \to a$. If $f \in D_A(x)$ then $f(a_n b) \to f(ab)$ and since $f(a_n b) \geqslant 0$, we have $f(ab) \geqslant 0$. This means that the element a belongs to $K(A)$ and $K(A)$ is closed.

It remains to see that $K(A)$ generates $H(A)$ if A has an approximate identity of norm less than or equal to 1. If A^+ is the unitization of A then $K(A^+)$ generates $H(A^+)$ and for h in $H(A)$, the following relation holds: $(0, h) = (\lambda_1, h_1) - (\lambda_2, h_2)$, $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$, and $h_1, h_2 \in K(A)$. Hence, $H(A) = K(A) - K(A)$. $\square$

Note 4.2. The condition in (I) is satisfied when A has an approximate bounded identity δ_n. By (I) we have

$$\sup_{\|x\|\leqslant 1} \|ax\| \leqslant \sup_{\|x\|\leqslant 1} \|a\| \cdot \|x\| = \|a\| = \Bigl\| \lim_{\lambda} a\delta_n \Bigr\| = \lim_{\lambda} \|a\delta_n\| \leqslant \sup_{\|x\|\leqslant 1} \|ax\|, \quad a \in A.$$

The next example exhibits that a Banach algebra with the norm (I) may not posses an approximate identity.

Example 4.3. Let D be the closed disc in the set of complex numbers C. Suppose $A(D)$ is the set of all functions f such that f is analytic on the interior of D with the usual pointwise operations and the sup norm $\|f\|_\infty = \sup_{\lambda \in D} |f(\lambda)|$, $f \in A(D)$. With this norm, $A(D)$ is a commutative Banach algebra.

Consider $A_0 = \{f \in A(D)\colon\ f(0) = 0\}$ and $g \in A_0$: $g(\lambda) = \lambda$. Then for $f \in A_0$ we have $\|f\| = \sup_D |f(\lambda)\lambda| = \sup |f(\lambda)g(\lambda)| = \|fg\|$. Suppose $\{\delta_n\}$ is a bounded approximate identity of A_0. Then, $\delta_n(0) = 0$. Define a linear functional ϕ as follows: $\phi(f) = f'(0)$, $\forall f \in A_0$. Here, $f'(0)$ is the first derivative of f at zero. Clearly, $\phi \in A_0$ and $\delta_n(\lambda)g(\lambda) \to g(\lambda)$, $\lambda \in D$. By taking the derivative of both sides we get $\delta_n'(\lambda)g(\lambda) + g'(\lambda)\delta(\lambda) \to g'(\lambda) = 1$. Hence, $\phi(\delta_n g) = \delta_n'(0)g(0) + g'(0)\delta_n(0) \to g'(0) = 1$. Observe that the left side is zero because $\delta_n(0) = 0 = g(0)$ while the right-hand side is equal to one. This leads to a mathematically false statement: $0 = 1$. Thus the algebra A_0 does not have an approximate identity.

In the next example we show that $H(A) = K(A) - K(A)$ even if the given norm is not equivalent to the norm given by (I).

Example 4.4. Consider the algebra ℓ_1 with point wise multiplication. Let $a = \left\{\frac{1}{n}, - - -, \frac{1}{n}, 0\right\}$ which is terminating after $n \geqslant 2$ entries. Then $\|a\|_1 = 1$. Also, $\|ax\|_1 \leqslant n^{-1}\|x\|_1$ and this can be arbitrarily small compared with $\|x\|_1$. Hence, the norms are not equivalent. The sets $H(\ell_1)$ and $K(\ell_1)$ are the real and positive sequences, respectively. This is possible because the numerical range lies between the convex hull of the range and its closure. Thus $H(A) = K(A) - K(A)$.

Note 4.5. We still do not know how much one can relax the requirements of the norm given by (I) to keep the cone $K(A)$ generating the real Banach space $H(A)$.

The next theorem has a different approach than the approach given in [12]. For C^*-algebra case, see [3, Theorem 12.3].

Theorem 4.6. *Let A be a B^*-algebra with the norm given in* (I). *Then $K(A)$ is proper, convex, closed, and normal cone in $H(A)$.*

Proof. Let $a \in K(A) \cap (K(-A))$. Then $V_A(a) \subseteq R^+$ and $-V_A(a) = V_A(-a) \subseteq R^+$. That is, $V_A(a) \subseteq R^+ \cap R^- = \{0\}$. Hence, $\{0\} = K(A) \cap (K(-A))$ because $a = 0$. □

Theorem 4.6 also shows that the ordering in the cone is antisymmetric.

Let $S(A)$ be the set of all self-adjoint elements of A. A partial ordering of $S(A)$ is induced by the positive cone $K(A)$ via the following order $(h_1 - h_2) \geqslant 0 \Leftrightarrow (h_1 - h_2) \in K(A)$. In [2] and [12] the positive cone is the closure of the set of finite sum of elements of the form (a^*a) and the involution is continuous. We consider the case when the involution is discontinuous.

Let $a \in K(A)$such that $a = a_1^*a_1 + - - - + a_n^*a_n$. We write the closure of $K(A)$ in $S(A)$ (not in A) as $\overline{K(A)}$. A linear functional ψ on A is positive if $\psi(a) \geqslant 0,\ \forall a \in A$.

Such a functional ψ is called a centralizer if $\psi(ab) = \psi(ba),\ \forall a, b \in A$. The set of all centralizers is denoted by $C(A)$. We recall that each positive multiplicative linear functional ψ is an α-bounded functional on A with $|\psi|_\alpha = 1$.

Let $\hat{A}$ be the set of all characters ϕ of A such that $\phi(a^*) = \overline{\phi(a)},\ \forall a \in A$. Let $P(A)$ be the set of all positive linear functional ψ on A with $\psi(e) = 1$. By the set $E(A)$ we mean that the set of all extreme points of $P(A)$.

We use our positive cone in a different role to derive a relation between the set $E(A)$ and the carrier space $\hat{A}$. It is not difficult to check that $\overline{K(A)} = S(A)$ if zero is the only positive linear functional on A.

We define the product between two elements a, b of A as $(a \circ b) = \frac{1}{2}(ab + ba)$. A cone K has the closure property if K is closed with respect to $(a \circ b)$.

In the following theorem we present a modified version of Theorem 2.1 of [12] which also improves Theorem 2 of [2].

Theorem 4.7. *If $\overline{K(A)}$ has the closure property then $E(A) = \hat{A}$.*

Proof. If the set $\overline{K(A)}$ has the closure property then we claim that each element of the set $E(A)$ is a nonzero multiplicative linear functional.

Let $\phi \in E(A)$. We must show that $\phi(ab) = \phi(a)\phi(b),\ \forall a \in A$, and a fixed b in $\overline{K(A)}$. Let $(e - b) \in \overline{K(A)}$. Then for all a in A the difference $[\phi(ab) - \phi(a)\phi(b)]$ exists. Moreover, the fixed b in $\overline{K(A)}$ allows us to assume that $\phi(ab) - \phi(a)\phi(b) = \eta_b(a)$. It should be noted that if $a = e$ then $0 = \eta_b(e)$.

This is the plan we have: we must find two positive linear functionals, say, ψ and ω such that $\psi \circ \omega = \phi$, where $\phi \in E(A)$. Once this is done then $0 = \eta_b(e)$ is automatic and in this case we will have $\phi(ab) - \phi(a)\phi(b) = 0$ for all elements a in A, b in $\overline{K(A)}$, and $(e - b) \in \overline{K(A)}$.

The construction of ψ and ω utilizes the definition of $0 = \eta_b(a)$. If $\psi(a) - \phi(a) = \eta_b(a) = \phi(a) - \omega(a)$ then $(\psi \circ \omega)$ exists and $(\psi \circ \omega) = \phi(a) = \frac{\psi(a)+\omega(a)}{2}$.

Next, let x be any element of $\overline{K(A)}$. Then $(x - xb) = x(e - b)$ also belongs to $\overline{K(A)}$ and

$$\psi(x) - \phi(xb) = \phi(x) - \phi(x)\phi(b) \quad \text{or} \quad \psi(x) = \phi(xb) - \phi(x)\phi(b) + \phi(x) \quad \text{or}$$
$$\psi(x) = \eta_b(x) + \phi(x).$$

Hence, $\psi(x) \geqslant 0$. Similarly $\omega(x) \geqslant 0$. Thus both the functionals ψ and ω belong to $P(A)$ and their product is defined and given by ϕ. The equality $0 = \eta_b(\cdot)$ now holds.

Also, let k be a real number such that $(e - kp) \in \overline{K(A)}$ for an arbitrarily chosen p in $\overline{K(A)}$. This is possible because the identity e of A is an interior point of $\overline{K(A)}$. But $S(A) = \overline{K(A)} - \overline{K(A)}$. Also, $A = S(A) + iS(A)$ implies that $0 = \eta_b(\cdot)$ holds for all a, b in A. Theorem 2 of [10] asserts that the Jordon homomorphism of a ring into an integral domain is a homomorphism or an anti-homomorphism and hence the functional ϕ is multiplicative and the proof is complete. □

Remark 4.8. The proof of Theorem 4.7 uses the definition of extreme points in full force. It suggests that each multiplicative functional is in $E(A)$. To see this, let $0 < t < 1$ and $u, v \in P(A)$. Then by convexity, $tu + (1-t)v \in \hat{A}$. Hence, there exists a multiplicative functional z with $z = tu + (1-t)v$. The following idea is drawn from Theorem 5.3 of [12] and Lemma 4 in [2]. That is, if s is a self-adjoint element of A then $(u(s))^2 \leqslant u(s^2)$ and $(v(s))^2 \leqslant v(s^2)$. By operating z on s^2 we have the equality $z(s^2) = (z(s))^2$. Now we claim that $u(s)v(s) \geqslant 0$, $\forall s \in S(A) = K(A)$. For, $z(s^2) = tu(s^2) + (1-t)v(s^2) \leqslant (tu(s^2) + (1-t)v(s^2))^2$. This means that $u(s)v(s) \geqslant u(s^2)v(s^2)$. Hence for all s in $S(A) = H(A)$, $u(s)v(s) \geqslant 0$. Suppose that for some $r \in S(A) = H(A)$, $u(r) \leqslant v(r)$, and for a real number k, $(ek + r) \in S(A)$ since $u(e) = 1 = v(e)$. Also, $u(ek + r) = k + u(r)$ implies that $u(ke + r) < k + v(r) < 0$. That is, $u(ke + r) < 0$. Similarly $v(ke + r) > 0$ and together these inequalities produce a contradiction. Thus u and v have to be equal.

We note that if $E(A) = \hat{A}$ then the Krein–Milman theorem guarantees that each $\psi \in P(A)$ also belongs to $C(A)$, the set of centralizers. This is true because each positive functional belongs to $C(A)$ as the weak $*$-limit of a finite series in the dual space of A. Moreover, every pure state of A is a hermitian complex homomorphism and hence Theorem 4.7 implies that every pure state of A is actually an extreme point of set of all positive linear functionals on A.

Theorem 4.9. *With the containment $P(A) \subseteq C(A)$, the cone $\overline{K(A)}$ has the closure property.*

Proof. Suppose a positive linear functional $\psi \in P(A)$ is a centralizer. That is, $\psi \in C(A)$ s.t. $\forall a, b \in A$, $\psi(ab) = \psi(ba)$.

If $a, b \in K(A)$ then $a = a_1^* a_1 + - - - + a_n^* a_n$ and $b = b_1^* b_1 + - - - + b_m^* b_m$. Also,

$$(a \circ b) = \left(\frac{ab + ba}{2}\right) = \frac{1}{2}\big[\big(a_1^* a_1 + - - - + a_n^* a_n\big)\big(b_1^* b_1 + - - - + b_m^* b_m\big)\big].$$

The above formula follows the following array of correspondence of a_i's and b_j's:

$$\begin{array}{c|c|c} a_1 & \to & b_1 \\ \hline a_2 & \to & b_2 \\ \hline \vdots & \vdots & \vdots \\ \hline a_{n-1} & \to & b_{m-1} \\ \hline a_n & \to & b_m \end{array} \quad \text{and} \quad a_1 \to b_2, - - -, a_{n-1} \to b_m.$$

A translation of the above diagram is given below:

$$\begin{aligned}(a \circ b) = {} & \big(a_1^* a_1\big) \circ \big(b_1^* b_1\big) + - - + \big(a_{n-1}^* a_{n-1}\big) \circ \big(b_{m-1}^* b_{m-1}\big) + - - \\ & + \big(a_n^* a_n\big) \circ \big(b_m^* b_m\big) + - - + \big(a_{n-1}^* a_{n-1}\big) \circ \big(b_m^* b_m\big).\end{aligned}$$

By the pattern above, one can conclude that the general form of $(a \circ b)$ is the finite sum of the form $(a_n^* a_n b_m^* b_m + b_m^* b_m a_n^* a_n)$. Let $\psi((a \circ b)) = \psi((a_n^* a_n) \circ (b_m^* b_m))$. Then without any loss of generality we have

$$\psi\big((a \circ b)\big) = \psi\big(\big(a_n^* a_n\big) \circ \big(b_m^* b_m\big)\big) = \psi\big((a^* a) \circ (b^* b)\big) = \psi(a^* a b^* b + b^* b a^* a).$$

Now $\psi \in C(A)$ yields the following identity:

$$\begin{aligned}\psi\big((a \circ b)\big) &= 2^{-1}\psi(a^* a b^* b + b^* b a^* a) = 2^{-1}\psi(a^* a b^* b) + 2^{-1}\psi(b^* b a^* a)\\ &= 2^{-1}\big(2\psi(b^* b a^* a)\big) = \psi(b^* b a^* a).\end{aligned}$$

That is,

$$\psi(b^* b a^* a) = \psi\big(b^*(b a^* a)\big) = \psi\big((b a^*)(a b^*)\big) = \psi\big((a b^*)(b a^*)\big).$$

Hence,

$$\psi(b^* b a^* a) = \psi\big((a b^*)(b a^*)\big) \geqslant 0.$$

Thus $\psi((a \circ b)) \geqslant 0$ and by the continuity of ψ and since it is arbitrarily chosen, it follows that for any elements c and d in $\overline{K(A)}$, the cone $\overline{K(A)}$ has the closure property. The last assertion is an outcome of Lemma 1.2 of [12]. □

Remark 4.10. Theorems 4.7 and 4.9 together provide with a characterization of the cone $\overline{K(A)}$ in terms of the centralizers.

5. An application of the Shirali–Ford theorem

The following definitions and preliminaries can be found in Doran and Belfi's book in [3], a rich and excellent source of these ideas. The $*$-involution in a $*$-algebra A is hermitian if every h in $H(A)$ has real spectrum and real numerical range. When the involution in a $*$-algebra A is hermitian then A is called a hermitian algebra.

The involution in a $*$-algebra A is said to be symmetric if every element of the form $(-x^* x)$ is quasi-regular in A (an element a is quasi-regular if there exists b in A such that $a + b - ab = b + a - ba$). When the involution in A is symmetric then A is called a symmetric algebra. If A has unit e, then A is symmetric if and only if $(e + a^* a)$ is invertible for all elements a in A. For details about the Shirali–Ford theorem we refer the reader to [1, p. 226] and [5]. The article by Kelly and Vaught in [12] has provided insight about this idea.

Let $L(A)$ be a positive cone such that if s belongs to $S(A)$ then $s = s_1^2 + - - - + s_m^2$. Let us assume that $\overline{L(A)}$ be the closure of $L(A)$ in $S(A)$ not in A so that the involution is not continuous.

Theorem 5.1. *For a symmetric algebra A, $\overline{L(A)} = \overline{K(A)}$.*

Proof. Let A be symmetric and $a \in A$. Then by [3, Theorem 33.2, p. 129], the spectrum $\sigma(a^* a) \geqslant 0$. Also, the Shirali–Ford theorem guarantees that $\sigma(a^* a) \subseteq R^+$. This argument can be extended to show that if $\upsilon > 0$ then $\sigma(e\upsilon + a^* a) \subseteq (0, \infty)$. It is not difficult to show that if $\sigma(a) \subseteq (0, \infty)$ then the element a can be written as an exponent of an element b in $S(A) = H(A)$ such that $b^2 = a = b^3 = - - -$. Hence, $\sigma(a) \subseteq (0, \infty)$ with $b = (e\upsilon + a^* a)^{1/2}$ and the cones $\overline{L(A)}$ and $\overline{K(A)}$ are equal. We would like to mention that if h is in $H(A) = S(A)$ and A is a C^*-algebra such that $\sigma(h) \subseteq R^+$ then $a^2 = h$, $a \in A$. □

The following example illustrates Theorem 5.1. It is shown that it is possible to have two identical cones but $0=(e+a^*a)$. More details on this idea are given in [11].

Example 5.2. Let M be the algebra of all 2 by 2 matrices. The involution is defined as follows:

$$\text{If} \quad A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{then} \quad A^*=\begin{pmatrix} \bar{0} & -\bar{1} \\ \bar{1} & \bar{0} \end{pmatrix}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Clearly, $A^*A=AA^*$ and if I is the 2 by 2 identity operator then

$$A^*A+I=AA^*+I=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

although the cones are equal by Theorem 5.1.

The above example is promising since it suggests that if $0=(e+A^*A)$ then cones in $S(A)=H(A)$ are identical. We believe that the inequality $\psi(e)\psi(a^2)\leqslant|\psi(a)|^2$, $\forall a\in S(A)$ is true and thus, $\psi(e)=0$. So in fact, one of the cones coincides with $S(A)$ and hence cones $\overline{L(A)}$ and $\overline{K(A)}$ are identical. We also note that if the algebra A is commutative then both the closed cones $\overline{L(A)}$ and $\overline{K(A)}$ are equal.

6. Cones versus Rad(A)

By Rad(A) we mean the Jacobson radical of algebra A, that is, the intersection of the kernels of all irreducible representations of A or the intersection of all the maximal left ideals of A. If A is a $*$-algebra then the $*$-radical of A is denoted by $R^*(A)$. We remark that for a Banach $*$-algebra, $R^*(A)$ is a closed $*$-ideal and $R^*(A)\supseteq \operatorname{Rad}(A)$.

If $a_1,---,a_n$ are elements of A then the Harte's spectrum $H(a_1,---,a_n)$ is defined to be the union of left and right joint spectra. Let H_L and H_R be the left and the right joint spectra, respectively. Then $H_L\cup H_R=H$. Here,

$$H_L=\left\{(\lambda_1,---,\lambda_n)\in C^n\colon \sum_{i=1}^{n} A(a_i-\lambda_i)\neq A\right\} \quad \text{and}$$

$$H_R=\left\{(\lambda_1,---,\lambda_n)\in C^n\colon \sum_{i=1}^{n} (a_i-\lambda_i)A\neq A\right\}.$$

For a single element a in A, $H(a)=\sigma(a)$. Recently the author in [6, Theorem 1.2] proved an improved version of his previous result in [7, Theorem 2] regarding these spectra. Here we present the following:

Theorem 6.1. *If $A/\operatorname{Rad}(A)$ is commutative then $H_L=H_R$, $\forall a_1,---,a_n\in A$.*

Proof. The proof is trivial and actually can be derived from Theorem 2 of [7] and Theorem 1.2 of [6].

Theorem 2 of [7] shows that the converse of Theorem 6.1 is true, since H_L and H_R are the same as of the spectrum of $(a_1+\operatorname{Rad}(A),---,a_n+\operatorname{Rad}(A))$. $\square$

The following result is an easy consequence of the above discussion.

Theorem 6.2. *If $H_L(a_1, ---, a_n) \subset H_R(a_1, ---, a_n)$ for all n-tuples $(a_1, ---, a_n) \in A^n$, $n = 1, 2, --$, then $H_L = H_R$ for arbitrary $a_1, ---, a_n \in A$, $n = 1, 2, ---$.*

Note 6.3. Theorem 6.2 raises a question about 'finite n'. In other words, does n belong to an interval of positive integers, say, for example, $[1, n_0]$, $n_0 > 1$? A paper containing the answer to this question is in preparation.

Theorem 6.4. *The cone $\overline{K(A)}$ has the closure property if and only if the algebra A is commutative modulo $R^*(A)$.*

Proof. Theorems 4.7 and 4.9 validate that the cone $\overline{K(A)}$ has the closure property if and only if each functional $\psi \in P(A)$ is also a centralizer, that is, $\psi \in C(A)$. Hence by the definition of centralizers we have $(ab - ba) \in \{\psi^{-1}(0)\colon\ \psi \in P(A),\ a, b \in A\}$. Thus by [3, Proposition 30.4, p. 120], $(ab - ba) \in R^*(A)$ and the proof is complete. □

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